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Random Perturbations of Axiom A Basic Sets

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In this paper we study small, random, diffeomorphism-type perturbations of an Axiom A basic set. By means of the structural stability of such a basic set with respect to time-dependent perturbations and by means of the Markov partition of the basic set, we apply the thermodynamic formalism of random subshifts of finite type to this situation, obtaining some ergodic-theoretic results concerning equilibrium states.

KEY WORDS: Axiom A basic set; bundle random dynamical system; equilibrium state; SRB measure.

INTRODUCTION

In this paper we consider random dynamical systems (RDS) generated by small, random, diffeomorphism-type perturbations of an Axiom A basic set of a deterministic diffeomorphism. We are concerned with the dynamical properties of such an RDS near the basic set. Each individual realization of the RDS is a time-dependent perturbation of the basic set. We first discuss the structural stability of a hyperbolic set with respect to such time-dependent perturbations. Using the structural stability results and the Markov partition method, we can give a symbolic representation of the RDS described above when the hyperbolic set is an Axiom A basic set (i.e., a locally maximal hyperbolic set which is transitive). This enables us to apply the thermodynamic formalism of random subshifts of finite type developed by Bogenschütz and Gundlach[7] and Gundlach[10] to the RDS, obtaining existence and uniqueness of equilibrium states for some suitable functions. This can be regarded as a partial version for the RDS described

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above of the program of applying statistical mechanics to diffeomorphisms presented by Bowen[8] for the deterministic case. For the original work on this program we refer the reader to Sinai[23] (for Anosov diffeomorphisms) and to Ruelle[19, 20] (for Axiom A attractors and for the formalism of equilibrium states). See also the references cited in ref. 8 for other earlier contributions to this subject.

In what follows we give some related definitions and a more precise formulation of our setup. Let M be a Riemannian manifold without boundary, O an open subset of M with compact closure, and $f: O \to M$ a C^r ($r \geqslant 1$) diffeomorphism to the image.

Let $\Lambda_0 \subset O$ be a compact set which is f-invariant, i.e., $f\Lambda_0 = \Lambda_0$. It is said to be *hyperbolic* if there is a continuous Tf-invariant splitting

$$T_{\Lambda_0} M = E^s \oplus E^u$$

and if there are two constants $0 < \lambda_0 < 1$ and $C > 0$ such that for all $n \geqslant 0$

$$|Tf^n \xi| \leqslant C\lambda_0^n \, |\xi| \qquad \text{for} \quad \xi \in E^s$$

$$|Tf^n \eta| \geqslant C^{-1}\lambda_0^{-n} \, |\eta| \qquad \text{for} \quad \eta \in E^u$$

Via a change of Riemannian metric we may—and will—always assume that $C = 1$. If Λ_0 is hyperbolic and moreover there exists a neighborhood U of Λ_0 such that $\bigcap_{n=-\infty}^{+\infty} f^n U = \Lambda_0$, then it is called a *locally maximal hyperbolic set* (LMHS) of f. An *Axiom A basic set* of f is defined to be an LMHS of f on which f is topologically transitive (i.e., f has a dense orbit).

By $C^r(O, M)$ ($r \geqslant 1$) we denote the set of all C^r maps from O to M equipped with the compact-open topology, which makes $C^r(O, M)$ a Polish space. Let $\mathrm{Emb}^r(O, M)$ be the Borel subset of $C^r(O, M)$ whose elements are diffeomorphisms from O to the images (i.e., embeddings). By $\mathscr{U}(f)$ we will always denote an open neighborhood of f in $\mathrm{Emb}^r(O, M)$ and, when it is given, we put

$$\Omega = \prod_{-\infty}^{+\infty} \mathscr{U}(f)$$

and let it have the product topology. For each $\omega \in \Omega$, we write $\omega = (..., g_{-1}(\omega), g_0(\omega), g_1(\omega),...)$ to express the sequence of maps corresponding to ω and let

$$g_\omega^n = \begin{cases} g_{n-1}(\omega) \circ \cdots \circ g_0(\omega) & \text{if} \quad n > 0 \\ \mathrm{id} & \text{if} \quad n = 0 \\ g_n(\omega)^{-1} \circ \cdots \circ g_{-1}(\omega)^{-1} & \text{if} \quad n < 0 \end{cases}$$

defined wherever they make sense. Denote by τ the left shift operator on Ω. When Ω is given, we will assume that P is a Borel probability distribution on Ω which is invariant and ergodic with respect to τ. In the rest of this paper, we will denote by $\mathscr{X}(P)$ the RDS generated by g_ω^n, $n \in \mathbf{Z}$, $\omega \in \Omega$, with ω being distributed according to P. Here we refer the reader to Arnold[1] for a general theory of RDSs. Clearly, each $\omega \in \Omega$ can be viewed as a time-dependent perturbation of f. In this paper we are concerned with the dynamical properties of such perturbations near a hyperbolic set of f when the neighborhood $\mathscr{U}(f)$ is sufficiently small.

Remark. Let O' be an open subset of O with $\overline{O'} \subset O$. If one takes $\mathscr{U}(f)$ as an open neighborhood of f in $C^r(O, M)$ and if it is sufficiently small, then $g|_{O'}: O' \to g(O')$ is a diffeomorphism for any $g \in \mathscr{U}(f)$ and one can define g_ω^n wherever they make sense in O'. With this modification of the definition of g_ω^n, all results given in this paper hold true if one takes O' as an open neighborhood of an Axiom A basic set of f and takes $\mathscr{U}(f)$ as a corresponding neighborhood of f in $C^r(O, M)$.

1. STRUCTURAL STABILITY OF HYPERBOLIC SETS

In this section we give some results on the structural stability of hyperbolic sets with respect to time-dependent perturbations (see Ruelle[22] for earlier discussion on this topic). In this section, except when indicated otherwise, we always assume $r = 1$ and so $\mathscr{U}(f)$ will be taken to be an open neighborhood of f in $\mathrm{Emb}^1(O, M)$, and we always assume that Λ_0 is a hyperbolic set of f.

Theorem 1.1. There exist a number $\varepsilon_0 > 0$ and an open neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^1(O, M)$ such that the following hold true:

(1) For each $\omega \in \Omega$ and any $x \in \Lambda_0$ there exists a unique point $x_\omega \in O$ such that $g_\omega^n x_\omega$ is well defined and

$$d(f^n x, g_\omega^n x_\omega) \leqslant \varepsilon_0 \tag{1.1}$$

for all $n \in \mathbf{Z}$.

(2) For any given $0 < \varepsilon \leqslant \varepsilon_0$ one can shrink $\mathscr{U}(f)$ given above so that (1) holds true with ε_0 being replaced with ε.

(3) Let $\omega \in \Omega$. Define $\Lambda_\omega = \{x_\omega : x \in \Lambda_0\}$ and

$$h_\omega : \Lambda_0 \to \Lambda_\omega, \qquad x \mapsto x_\omega$$

Then Λ_ω is compact and h_ω is a homeomorphism for all $\omega \in \Omega$. Moreover, the family of maps $\{h_\omega\}_{\omega \in \Omega}$ has the following properties:

(i) $g_0(\omega)\Lambda_\omega = \Lambda_{\tau\omega}$, $h_{\tau\omega}\circ f = g_0(\omega)\circ h_\omega$ for all $\omega\in\Omega$.

(ii) $\{h_\omega\}_{\omega\in\Omega}$ is equi-continuous in the sense that for any given $\varepsilon'>0$ one can find $\delta>0$ such that $d(x, y)<\delta$ implies $d(h_\omega x, h_\omega y)<\varepsilon'$ for any x, $y\in\Lambda_0$ and any $\omega\in\Omega$. So is the family $\{h_\omega^{-1}\}_{\omega\in\Omega}$ in an analogous sense.

(iii) The map $H\colon \Omega\to C^0(\Lambda_0, M)$, $\omega\mapsto h_\omega$ is continuous.

Proof. (1), (2), and the first part of (3) can be proved by standard arguments in structural stability theory of hyperbolic dynamical systems and hence their proofs are omitted here. (3)(i) is a natural corollary of (1). (3)(iii) follows from (1) and (3)(ii) by the Arzela–Ascoli Lemma. Finally, (3)(ii) follows from Lemma 1.2, which is given just below and which will be also useful for later arguments. ∎

Lemma 1.2. Let Λ_0 be as given above. Then one can find a neighborhood U_0 of Λ_0, a neighborhood $\mathscr{U}_0(f)$ of f in $\mathrm{Emb}^1(O, M)$, and numbers $\rho_0>0$, $C_0>0$, $\alpha_0\in(0, 1)$ such that the following holds true: If $\omega\in\Omega_0 := \prod_{-\infty}^{+\infty}\mathscr{U}_0(f)$, x, $y\in U_0$, $g_\omega^n x$, $g_\omega^n y$ are well defined, $g_\omega^n x$, $g_\omega^n y\in U_0$, and $d(g_\omega^n x, g_\omega^n y)\leqslant\rho_0$ for $n\in[-N, N]$, then $d(x, y)\leqslant C_0\alpha_0^N$.

Proof. This result is an easy consequence of persistence of the hyperbolic structure of Λ_0 under small perturbations. It can be proved as follows. Take a neighborhood U_0 of Λ_0 and a neighborhood $\mathscr{U}_0(f)$ of f in $\mathrm{Emb}^1(O, M)$ which have the following properties:

(i) There exists an extension of $T_{\Lambda_0}M = E^s\oplus E^u$ to a continuous splitting $T_{U_0}M = E^1\oplus E^2$ and there is a number $C>0$ such that for any $\xi=\xi^1+\xi^2\in E^1\oplus E^2$ one has

$$\|\xi\|_0 := \max\{|\xi^1|, |\xi^2|\}\leqslant C|\xi|$$

(ii) There exist positive numbers ρ_0, a_0, and ε_0 with $\lambda_0<a_0<1$ (λ_0 is the hyperbolic number of Λ_0) and $0<\varepsilon_0<\min\{\frac{1}{2}(1-a_0), \frac{1}{2}(a_0^{-1}-1)\}$ such that the following hold true: If $x\in U_0$, $g\in\mathscr{U}_0(f)$, and $gx\in U_0$, then

$$G_{g,x} := \exp_{g(x)}^{-1}\circ g\circ\exp_x\colon \{\xi\in T_xM : |\xi|\leqslant\rho_0\}\to T_{g(x)}M$$

is well defined and, writing

$$T_0G_{g,x} = \begin{pmatrix} A_{g,x} & C_{g,x} \\ D_{g,x} & B_{g,x}\end{pmatrix}\colon E_x^1\oplus E_x^2\to E_{g(x)}^1\oplus E_{g(x)}^2$$

we can express $G_{g,x}$ in the following way:

$$G_{g,x} = \begin{pmatrix} A_{g,x} & 0 \\ 0 & B_{g,x}\end{pmatrix} + R_{g,x}(\cdot)$$

where the linear maps $A_{g,x}: E_x^1 \to E_{g(x)}^1$, $B_{g,x}: E_x^2 \to E_{g(x)}^2$ satisfy $|A_{g,x}| \leq a_0$, $|B_{g,x}^{-1}| \leq a_0$ and where $R_{g,x}(\cdot)$ is a Lipschitz map whose Lipschitz constant with respect to $\|\cdot\|_0$ is not bigger than ε_0; moreover, the map

$$G_{g,x}^- := \exp_x^{-1} \circ g^{-1} \circ \exp_{g(x)}: \quad \{\xi \in T_{g(x)}M : |\xi| \leq \rho_0\} \to T_x M$$

is also well defined and has similar properties.

Now, if $\omega \in \Omega_0$ and $x, y \in U_0$ are as given in the formulation of the lemma, putting $\exp_x^{-1} y = \xi = \xi^1 + \xi^2 \in E^1 \oplus E^2$ and assuming without loss of generality $|\xi^2| \geq |\xi^1|$, then we can easily see that

$$C\rho_0 \geq \|\exp_{g_\omega^N x}^{-1} g_\omega^N y\|_0 \geq (a_0^{-1} - \varepsilon_0)^N \|\xi\|_0$$

which implies

$$d(x, y) = |\xi| \leq 2C\rho_0 (a_0^{-1} - \varepsilon_0)^{-N}$$

Taking $C_0 = 2C\rho_0$ and $\alpha_0 = (a_0^{-1} - \varepsilon_0)^{-1}$, we complete the proof. ∎

The next result extends Nitecki's result (ref. 16, Proposition 3) on C^1 upper semistability of neighborhoods of hyperbolic sets to the case of time-dependent perturbations. The proof goes along the same line as that of ref. 16, Proposition 3, and will be omitted here. This result will be useful in Section 3.3, where we will deal with the ergodic theory of random perturbations of hyperbolic attractors.

Proposition 1.3. There exists a neighborhood V_0 of Λ_0 and for any given $\varepsilon > 0$ one can find a neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^1(O, M)$ such that there exists a family of continuous maps $\{H_\omega: V_0 \to V_\omega := H_\omega(V_0)\}_{\omega \in \Omega}$ which makes the following diagram commutative:

$$\begin{array}{ccc} V_0 \cap f^{-1} V_0 & \xrightarrow{f} & V_0 \\ \downarrow H_\omega & & \downarrow H_{\tau\omega} \\ V_\omega \cap g_0(\omega)^{-1} V_{\tau\omega} & \xrightarrow{g_0(\omega)} & V_{\tau\omega} \end{array}$$

and satisfies $d(H_\omega, \mathrm{id}) < \varepsilon$ *for each* $\omega \in \Omega$.

Remark 1.4. Let ε and $\mathscr{U}(f)$ be as given in Proposition 1.3. If $\varepsilon \leq \varepsilon_0$ (ε_0 is as given in Theorem 1.1) and $\mathscr{U}(f)$ also satisfies Theorem 1.1 (2), then clearly $h_\omega = H_\omega|_{\Lambda_0}$ for all $\omega \in \Omega$.

The next proposition is a result describing the stability of the hyperbolic splitting of Λ_0 with respect to time-dependent perturbations. Let $\{\Lambda_\omega\}_{\omega \in \Omega}$ be as introduced in Theorem 1.1. Put

$$\Lambda = \bigcup_{\omega \in \Omega} \{\omega\} \times \Lambda_\omega$$

and define

$$G\colon \quad \Lambda \to \Lambda, \qquad (\omega, x) \mapsto (\tau\omega, g_0(\omega)x)$$

Let E_Λ be the pullback of TM by means of the projection $p_2\colon \Lambda \to M$, $(\omega, x) \mapsto x$. Define

$$\gamma_0 = \inf_{x \in \Lambda_0} \gamma(E^s_x, E^u_x)$$

where $\gamma(\cdot, \cdot)$ denotes the angle between the two associated spaces.

Proposition 1.5. For any given $\lambda \in (\lambda_0, 1)$ and $\gamma \in (0, \gamma_0)$ one can take $\mathscr{U}(f)$ small enough so that the following hold true:

(1) There is a continuous splitting $E_\Lambda = E^s_\Lambda \oplus E^u_\Lambda$ such that for each $(\omega, x) \in \Lambda$

$$T_x g_0(\omega)\, E^s_{(\omega, x)} = E^s_{G(\omega, x)}, \qquad T_x g_0(\omega)\, E^u_{(\omega, x)} = E^u_{G(\omega, x)}$$

and

$$|T_x g_0(\omega)\, \xi| \leqslant \lambda\, |\xi| \qquad \text{for} \quad \xi \in E^s_{(\omega, x)}$$
$$|T_x g_0(\omega)\, \eta| \geqslant \lambda^{-1}\, |\eta| \qquad \text{for} \quad \eta \in E^u_{(\omega, x)}$$

(2) $\gamma(E^s_{(\omega, x)}, E^u_{(\omega, x)}) \geqslant \gamma$ for all $(\omega, x) \in \Lambda$.

(3) With λ and γ given above, if f is of class C^2, one can find a neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ such that the splitting $E^s_\Lambda \oplus E^u_\Lambda$ is equi-Hölder continuous in the following sense: There exist constants $C > 0$ and $\theta > 0$ [depending only on $(\lambda, \gamma, \mathscr{U}(f))$] such that

$$d(E^a_{(\omega, x)}, E^a_{(\omega, y)}) \leqslant C d(x, y)^\theta, \qquad a = s, u$$

for all $x, y \in \Lambda_\omega$ and any $\omega \in \Omega$.

Proof. The proof of (1) and (2) is a standard argument (see, for instance, the proof of Liu and Qian ref. 13, Proposition VII.2.1, with a slight modification). The proof of (3) is the same as that of ref. 13, Proposition VII.2.4. ∎

We end this section with a result which is irrelevant to the later arguments. It describes the stability of a hyperbolic set with respect to C^0-small time-dependent perturbations. As before, let Λ_0 be a hyperbolic set of $f \in \mathrm{Emb}^1(O, M)$. Define $H^0(O, M) = \{g: O \to M, g$ is a homeomorphism from O to the image$\}$ and endow it with the C^0 compact-open topology.

Theorem 1.6. For any given $\varepsilon > 0$, there exists a neighborhood $\mathscr{U}(f)$ of f in $H^0(O, M)$ such that there exist a family of compact sets $\{\Lambda'_\omega\}_{\omega \in \Omega}$ $[\Omega = \prod_{-\infty}^{+\infty} \mathscr{U}(f)]$ and a family of surjective continuous maps $\{h'_\omega: \Lambda'_\omega \to \Lambda_0\}_{\omega \in \Omega}$ which satisfy $d(h'_\omega, \mathrm{id}) < \varepsilon$, $g_0(\omega)\, \Lambda'_\omega = \Lambda'_{\tau\omega}$ and make the following diagram commutative:

$$\begin{array}{ccc} \Lambda_0 & \xrightarrow{\ f\ } & \Lambda_0 \\ \uparrow{\scriptstyle h'_\omega} & & \uparrow{\scriptstyle h'_{\tau\omega}} \\ \Lambda'_\omega & \xrightarrow{\ g_0(\omega)\ } & \Lambda'_{\tau\omega} \end{array}$$

for each $\omega \in \Omega$.

The proof of this theorem is almost the same as that of Nitecki ref. 16, Proposition 1.

2. EQUILIBRIUM STATES OF GENERAL EQUI-HÖLDER CONTINUOUS FUNCTIONS

2.1. Basic Strategy and Basic Notions

In this section we consider equilibrium states of general equi-Hölder continuous functions for random perturbations of Axiom A basic sets as described at the beginning of the paper. We will assume $r = 1$ throughout this section. Let Λ_0 be an Axiom A basic set of f, and let $\mathscr{U}(f)$ and $\Lambda = \bigcup_{\omega \in \Omega} \{\omega\} \times \Lambda_\omega$ be as given in Theorem 1.1. Write

$$\mathscr{G} = \{g^n_\omega: \Lambda_\omega \to \Lambda_{\tau^n\omega}, n \in \mathbf{Z}, \omega \in \Omega\}$$

It is a *bundle RDS* over $(\Omega, \mathscr{B}(\Omega), P, \tau)$ with $G: \Lambda \to \Lambda$ being the corresponding skew-product transformation, where $\mathscr{B}(\Omega)$ is the Borel σ-algebra of Ω. (See Bogenschütz and Gundlach[7] or Bogenschütz[6] for a general theory of bundle RDSs.) Our basic strategy is as follows. By means of a Markov partition of Λ_0 and the family of homeomorphisms $\{h_\omega\}_{\omega \in \Omega}$ we will obtain a simple symbolic representation of the bundle RDS $\mathscr{G}$. This will allow us to apply the thermodynamic formalism for random subshifts of finite type, developed in Bogenschütz and Gundlach[7] and Gundlach,[10] to $\mathscr{G}$, obtaining existence and uniqueness of equilibrium states of $\mathscr{G}$ for some suitable functions $\varphi: \Lambda \to \mathbf{R}$.

Before going to our main result of this section, we first review briefly in this subsection the notions of pressure and equilibrium states for the bundle RDS $\mathscr{G}$. In what follows we assume that the σ-algebra on Ω is the completion of $\mathscr{B}(\Omega)$ with respect to P, written $\bar{\mathscr{B}}(\Omega)$. Writing $X = \bar{O}$ and following ref. 7, we denote by $L^1_\Lambda(\Omega, C(X))$ the collection of all families $\varphi = \{\varphi_\omega \in C(\Lambda_\omega)\}_{\omega \in \Omega}$ which are such that $(\omega, x) \mapsto \varphi_\omega(x)$ is measurable on Λ and $\|\varphi\| := \int \sup_{x \in \Lambda_\omega} |\varphi_\omega(x)| \, dP(\omega) < +\infty$. With respect to the norm $\|\cdot\|$, $L^1_\Lambda(\Omega, C(X))$ is a Banach space.[7]

Let $\varepsilon > 0$, $\omega \in \Omega$, and $n \geqslant 1$. A set $F \subset \Lambda_\omega$ is called *(ω, n, ε)-separated* if $\max\{d(g^k_\omega x, g^k_\omega y): 0 \leqslant k \leqslant n-1\} > \varepsilon$ for any x, $y \in F$ with $x \neq y$. Given $\varphi \in L^1_\Lambda(\Omega, C(X))$, we define for $n \geqslant 1$

$$(S_n \varphi)_\omega(x) = \sum_{i=0}^{n-1} \varphi_{\tau^i \omega}(g^i_\omega x)$$

and

$$\pi_{\mathscr{G}}(\varphi)(\omega, n, \varepsilon) = \sup\left\{\sum_{x \in F} \exp(S_n \varphi)_\omega(x): F \text{ is an } (\omega, n, \varepsilon)\text{-separated subset of } \Lambda_\omega\right\}$$

Then we call

$$\pi_{\mathscr{G}}(\varphi) := \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \int \log \pi_{\mathscr{G}}(\varphi)(\omega, n, \varepsilon) \, dP(\omega)$$

the *topological pressure* of φ with respect to $\mathscr{G}$.

Here we claim that, though it is not yet clear whether the topological pressure can be defined equivalently by using liminf instead of limsup for a general continuous compact bundle RDS (see Gundlach[10] and Meyer[15]), for our present bundle RDS $\mathscr{G}$ we have

$$\pi_{\mathscr{G}}(\varphi) = \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{n} \int \log \pi_{\mathscr{G}}(\varphi)(\omega, n, \varepsilon) \, dP(\omega)$$

for each $\varphi \in L^1_\Lambda(\Omega, C(X))$. Indeed, one can view $f: \Lambda_0 \to \Lambda_0$ as a special RDS $\mathscr{F}_0$ over $(\Omega, \bar{\mathscr{B}}(\Omega), P, \tau)$ and $\mathscr{G}$ is conjugate to $\mathscr{F}_0$ by means of $\{h_\omega\}_{\omega \in \Omega}$. Thus for every $\varphi \in L^1_\Lambda(\Omega, C(X))$ one has

$$\pi_{\mathscr{G}}(\varphi) = \pi_{\mathscr{F}_0}(\psi) \tag{2.1}$$

where $\psi = \{\psi_\omega = \varphi_\omega \circ h_\omega\}_{\omega \in \Omega}$. Along the line of Walters (ref. 24, Chap. 9), one can prove that

$$\pi_{\mathscr{F}_0}(\psi) = \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{n} \int \log \pi_{\mathscr{F}_0}(\psi)(\omega, n, \varepsilon)\, dP(\omega)$$

Since $\{h_\omega\}_{\omega \in \Omega}$ and $\{h_\omega^{-1}\}_{\omega \in \Omega}$ are equicontinuous, for any given $\varepsilon > 0$ one can find $\delta_\varepsilon > 0$ such that $\delta_\varepsilon \to 0$ when $\varepsilon \to 0$ and for each $\omega \in \Omega$, $x, y \in \Lambda_\omega$ with $d(x, y) \leqslant \delta_\varepsilon$ implies $d(h_\omega^{-1}x, h_\omega^{-1}y) \leqslant \varepsilon$. It is easy to see that

$$\pi_{\mathscr{F}_0}(\psi)(\omega, n, \varepsilon) \leqslant \pi_{\mathscr{G}}(\varphi)(\omega, n, \delta_\varepsilon)$$

for all $\omega \in \Omega$, $n \geqslant 1$, and $\varepsilon > 0$. This together with (2.1) proves what we claimed above. This fact will be useful in proving Proposition 3.1.

The *variational principle*[10] ensures that for $\varphi \in L^1_\Lambda(\Omega, C(X))$ one has

$$\pi_{\mathscr{G}}(\varphi) = \sup \left\{ h_\mu(\mathscr{G}) + \int \varphi\, d\mu : \mu \in \mathscr{M}(\Lambda, \mathscr{G}) \right\} \tag{2.2}$$

where $\mathscr{M}(\Lambda, \mathscr{G})$ denotes the set of $\mathscr{G}$-invariant measures (i.e., Borel probability measures on Λ, which is G-invariant and whose projection on Ω is P), $\int \varphi\, d\mu = \int \varphi_\omega(x)\, d\mu(\omega, x)$, and $h_\mu(\mathscr{G})$ denotes the entropy of $(\mathscr{G}, \mu)$ (see refs. 5 and 6 for a detailed treatment of the entropy theory of bundle RDSs). Actually, by the ergodic decomposition theorem,[6] (2.2) can be written as

$$\pi_{\mathscr{G}}(\varphi) = \sup \left\{ h_\mu(\mathscr{G}) + \int \varphi\, d\mu : \mu \in \mathscr{M}_e(\Lambda, \mathscr{G}) \right\}$$

where $\mathscr{M}_e(\Lambda, \mathscr{G})$ denotes the set of ergodic (with respect to G) elements of $\mathscr{M}(\Lambda, \mathscr{G})$. If $\mu \in \mathscr{M}(\Lambda, \mathscr{G})$ satisfies

$$\pi_{\mathscr{G}}(\varphi) = h_\mu(\mathscr{G}) + \int \varphi\, d\mu$$

then we call μ an *equilibrium state* of φ with respect to $\mathscr{G}$.

2.2. Thermodynamic Formalism of Subshifts of Finite Type with Random Potentials

If $A = (A_{ij})$ is an $n_0 \times n_0$ matrix with each entry being 0 or 1, let

$$\Sigma_A = \{\underline{a} = (a_i) \in \{1, \ldots, n_0\}^{\mathbf{Z}} : A_{a_i a_{i+1}} = 1, i \in \mathbf{Z}\}$$

We will always assume that each $k \in \{1,..., n_0\}$ can occur as a_0 for some $\underline{a} \in \Sigma_A$. Then the shift operator $\sigma: \Sigma_A \to \Sigma_A$ is called a subshift of finite type.

In this subsection we present some facts from the thermodynamic formalism for such a deterministic subshift operator $\sigma: \Sigma_A \to \Sigma_A$ with Σ_A having random potentials modeled over $(\Omega, \bar{\mathscr{B}}(\Omega), P, \tau)$. Here we refer the reader to refs. 7 and 10 for a theory of the thermodynamic formalism of general random subshifts of finite type; our present setup is a special situation of that general theory. By $L^1(\Omega, C(\Sigma_A))$ we denote the collection of all families $\psi = \{\psi_\omega \in C(\Sigma_A)\}_{\omega \in \Omega}$ which are such that $(\omega, \underline{a}) \mapsto \psi_\omega(\underline{a})$ is measurable and $\|\psi\| := \int \sup_{\underline{a} \in \Sigma_A} |\psi_\omega(\underline{a})| \, dP(\omega) < +\infty$. So by a function $\psi \in L^1(\Omega, C(\Sigma_A))$ we mean an integrable random potential of Σ_A. For $\psi \in L^1(\Omega, C(\Sigma_A))$, the pressure $\pi_\sigma(\psi)$ of ψ with respect to σ can be defined in a way similar to the definition of $\pi_{\mathscr{G}}(\varphi)$ given in the previous subsection. (See ref. 10 for various equivalent definitions of pressure for general random subshifts of finite type.)

We now apply the general theory developed in refs. 7 and 10 to our present situation to obtain some results concerning equilibrium states of random potentials for the shift $\sigma: \Sigma_A \to \Sigma_A$. For $\underline{a}, \underline{b} \in \Sigma_A$, we write $\underline{a} \overset{n}{\sim} \underline{b}$ if $a_i = b_i$ for all i with $|i| \leqslant n$. A function $\psi \in L^1(\Omega, C(\Sigma_A))$ is called *equi-Hölder continuous* if there exist $C > 0$ and $\alpha \in (0, 1)$ such that for P-a.e. ω one has

$$\operatorname{var}_n \psi_\omega := \sup\{|\psi_\omega(\underline{a}) - \psi_\omega(\underline{b})| : \underline{a}, \underline{b} \in \Sigma_A, \underline{a} \overset{n}{\sim} \underline{b}\} \leqslant C\alpha^n$$

for all $n \in \mathbf{Z}^+$. Define

$$\Theta: \Omega \times \Sigma_A \to \Omega \times \Sigma_A, (\omega, \underline{a}) \mapsto (\tau\omega, \sigma\underline{a})$$

and let $\mathscr{M}(\Omega \times \Sigma_A, \sigma)$ denote the set of Θ-invariant probability measures on $\Omega \times \Sigma_A$ which have marginal P on Ω. For $\mu \in \mathscr{M}(\Omega \times \Sigma_A, \sigma)$, let μ_ω, $\omega \in \Omega$ denote the conditional measures of μ on $\{\omega\} \times \Sigma_A$, which is identified with Σ_A, $\omega \in \Omega$. Then, according to refs. 7 and 10, we have the following results. If there exists $N \in \mathbf{N}$ such that $A^N > 0$ (i.e., A^N has no zero entries), then for each equi-Hölder continuous $\psi \in L^1(\Omega, C(\Sigma_A))$, there exists a unique $\mu_\psi \in \mathscr{M}(\Omega \times \Sigma_A, \sigma)$ such that

$$\pi_\sigma(\psi) = h_{\mu_\psi}(\sigma) + \int \psi \, d\mu_\psi$$

i.e., μ_ψ is a unique equilibrium state of ψ with respect to σ. This μ_ψ is ergodic (with respect to Θ) and is strong-mixing in the following sense:

Denoting by $L^2_A(\mu_\psi)$ the collection of all families $\zeta = \{\zeta_\omega : \Sigma_A \to \mathbf{R}\}_{\omega \in \Omega}$ which are such that $(\omega, \underline{a}) \mapsto \zeta_\omega(\underline{a})$ is measurable and $\zeta_\omega \in L^2((\mu_\psi)_\omega)$ for P-a.e. ω, then for any $\xi, \eta \in L^2_A(\mu_\psi)$ one has

$$\lim_{n \to +\infty} \left| \int (\xi_{\tau^n\omega} \circ \sigma^n)\, \eta_\omega \, d(\mu_\psi)_\omega - \int \xi_{\tau^n\omega}\, d(\mu_\psi)_{\tau^n\omega} \int \eta_\omega \, d(\mu_\psi)_\omega \right| = 0$$

[One can define such a strong-mixing property for the bundle RDS $(\mathscr{G}, \mu)$ described in the last subsection in a similar way (replacing Σ_A with Λ_ω).] Another main property of μ_ψ is that it is the thermodynamic limit of so-called Gibbs distributions on finite sequence spaces. Its conditional measures $(\mu_\psi)_\omega$, $\omega \in \Omega$ (called *sample Gibbs states*), have the following Gibbs property: There exist $C_1, C_2 > 0$ such that for P-a.e. ω

$$C_1 \leqslant \frac{(\mu_\psi)_\omega \{\underline{b} \in \Sigma_A : b_i = a_i \text{ for } i = 0,\ldots, m-1\}}{\exp[\sum_{i=0}^{m-1} \tilde{\psi}_{\tau^i\omega}(\sigma^i \underline{a}) - \log \lambda(\tau^{m-1}\omega) \cdots \lambda(\omega)]} \leqslant C_2 \tag{2.3}$$

for all $m \in \mathbf{N}$ and all $\underline{a} \in \Sigma_A$, where $\tilde{\psi}$ is a one-sided function [i.e., for each ω, $\tilde{\psi}_\omega(\underline{a}) = \tilde{\psi}_\omega(\underline{b})$ whenever $a_i = b_i$ for all $i \geqslant 0$] cohomologous to ψ [namely, there exist $u \in L^1(\Omega, C(\Sigma_A))$ and $c \in L^1(\Omega, P)$ such that $\tilde{\psi} = \psi + u - u \circ \Theta + c$], $\lambda(\omega)$ are random eigenvalues corresponding to $\tilde{\psi}$ given by Theorem 2.3 of ref. 10 (which is a random version of Ruelle's transfer operator theorem) and they satisfy $\int \log \lambda(\omega)\, dP(\omega) = \pi_\sigma(\psi)$.

2.3. Existence and Uniqueness of Equilibrium States

Let $L^1_A(\Omega, C(X))$ be as introduced in Section 2.1. A function $\varphi \in L^1_A(\Omega, C(X))$ is called *equi-Hölder continuous* if there exist constants $C > 0$ and $\theta > 0$ such that for P-a.e. ω one has $|\varphi_\omega(x) - \varphi_\omega(y)| \leqslant Cd(x, y)^\theta$ for any $x, y \in \Lambda_\omega$. The main result of this section is the following

Theorem 2.1. Let Λ_0 be a C^1 Axiom A basic set of f and let $\mathscr{U}(f)$ be an open neighborhood of f in $\mathrm{Emb}^1(O, M)$. If $\mathscr{U}(f)$ is given sufficiently small, then the corresponding bundle RDS $\mathscr{G}$ over $(\Omega, \bar{\mathscr{B}}(\Omega), P, \tau)$ has the following properties: For every equi-Hölder continuous function $\varphi \in L^1_A(\Omega, C(X))$, there exists a unique equilibrium state, written μ_φ, of φ with respect to $\mathscr{G}$. Furthermore, μ_φ is ergodic; and it is strong-mixing in the sense described in Section 2.2 if $f|_{\Lambda_0}$ is topologically mixing.

Proof. Let U_0, $\mathscr{U}_0(f)$, ρ_0, C_0, and α_0 be as introduced in Lemma 1.2. Let now $\mathscr{U}(f)$ be an open neighborhood of f in $\mathrm{Emb}^1(O, M)$ with $\mathscr{U}(f) \subset \mathscr{U}_0(f)$ such that Theorem 1.1 holds true for $\mathscr{U}(f)$ with $\Lambda_\omega \subset U_0$ and

$d(h_\omega, \mathrm{id}) < \rho_0/3$ for all $\omega \in \Omega$ [remember that $\Omega := \prod_{-\infty}^{+\infty} \mathscr{U}(f)$ when $\mathscr{U}(f)$ is given].

Let $\mathscr{R}$ be a Markov partition for (f, Λ_0) of diameter at most $\rho_0/3$, as constructed in Bowen, ref. 8, Section 3.C. Let the transition matrix A of $(f, \mathscr{R})$ and the map $\pi: \Sigma_A \to \Lambda_0$ be as defined in ref. 8, Section 3.D. Let $\sigma: \Sigma_A \to \Sigma_A$ be the subshift of finite type corresponding to A.

Now let $\mathscr{G}$ be the bundle RDS over $(\Omega, \bar{\mathscr{B}}(\Omega), P, \tau)$ corresponding to $\mathscr{U}(f)$. Define

$$\Pi: \Omega \times \Sigma_A \to \Lambda, \qquad (\omega, \underline{a}) \mapsto (\omega, h_\omega \pi \underline{a})$$

By the properties of π and Theorem 1.1, Π is a surjective continuous map, $\Pi \circ \Theta = G \circ \Pi$, and Π is one-to-one over the set $\Lambda \backslash \Delta$, where $\Delta = \bigcup_{\omega \in \Omega} \{\omega\} \times [h_\omega \bigcup_{j \in \mathbf{Z}} f^j(\partial^s \mathscr{R} \cup \partial^u \mathscr{R})]$ and $\partial^s \mathscr{R} \cup \partial^u \mathscr{R}$ is the boundary of $\mathscr{R}$ as defined in ref. 8.

Now let $\varphi \in L^1_\Lambda(\Omega, C(X))$ be an equi-Hölder continuous function with Hölder exponent $\theta > 0$ and constant $C > 0$. Define $\varphi^* = \varphi \circ \Pi$. Clearly $\varphi^* \in L^1(\Omega, C(\Sigma_A))$. Write $\varphi^* = \{\varphi^*_\omega \in C(\Sigma_A)\}_{\omega \in \Omega}$. Then, by Lemma 1.2, for P-a.e. ω one has

$$\mathrm{var}_n \, \varphi^*_\omega \leqslant C C_0^\theta (\alpha_0^\theta)^n$$

for all $n \in \mathbf{Z}^+$, that is, φ^* is equi-Hölder continuous.

We will assume that $f|_{\Lambda_0}$ is topologically mixing. Extension of the theorem to the general case is standard by the spectral decomposition theorem. Then $\sigma: \Sigma_A \to \Sigma_A$ is also topologically mixing, or equivalently, there is $N > 0$ such that $A^N > 0$. According to Section 2.2, there exists a unique equilibrium state, written $\mu_{\varphi^*} \in \mathscr{M}(\Omega \times \Sigma_A, \sigma)$, of φ^* with respect to σ. This μ_{φ^*} is ergodic and strong-mixing, and its conditional measures $(\mu_{\varphi^*})_\omega$ on $\{\omega\} \times \Sigma_A$ have the Gibbs property (2.3). We now prove that $\mu_{\varphi^*}(\Pi^{-1}\Delta) = 0$. Put $D_s = \pi^{-1}(\partial^s \mathscr{R})$ and $D_u = \pi^{-1}(\partial^u \mathscr{R})$. They are closed subsets of Σ_A, each smaller than Σ_A, and $\sigma D_s \subset D_s$, $\sigma^{-1} D_u \subset D_u$. Since μ_{φ^*} is Θ-invariant and ergodic, one has

$$\mu_{\varphi^*}(\Omega \times D_s) = \mu_{\varphi^*}\left[\bigcap_{n \geqslant 0} \Theta^n(\Omega \times D_s)\right] = 0 \quad \text{or } 1$$

Noting that the complement of D_s in Σ_A is a nonempty open set, by (2.3) we know that $\Sigma_A \backslash D_s$ has positive $(\mu_{\varphi^*})_\omega$ measure for P-a.e. ω and hence $\mu_{\varphi^*}(\Omega \times D_s) = 0$. Similarly one gets $\mu_{\varphi^*}(\Omega \times D_u) = 0$. This proves $\mu_{\varphi^*}(\Pi^{-1}\Delta) = 0$ since $\Pi^{-1}\Delta = \bigcup_{-\infty}^{+\infty} \Theta^n[\Omega \times (D_s \cup D_u)]$. Now let $\mu_\varphi = \Pi^* \mu_{\varphi^*}$, i.e., $\mu_\varphi(E) = \mu_{\varphi^*}(\Pi^{-1} E)$ for $E \in \mathscr{B}(\Lambda)$. Then μ_φ is clearly

G-invariant and its projection on Ω is P, hence it is an invariant measure of the bundle RDS $\mathscr{G}$, i.e., $\mu_\varphi \in \mathscr{M}(\Lambda, \mathscr{G})$. Since Π is an isomorphism between the two bundle RDSs $(\sigma, \mu_{\varphi^*})$ and $(\mathscr{G}, \mu_\varphi)$, we have $h_{\mu_{\varphi^*}}(\sigma) = h_{\mu_\varphi}(\mathscr{G})$. (See Bogenschütz[6] for the definition of isomorphisms between bundle RDSs and for the related results.) So

$$h_{\mu_\varphi}(\mathscr{G}) + \int \varphi \, d\mu_\varphi = h_{\mu_{\varphi^*}}(\sigma) + \int \varphi^* \, d\mu_{\varphi^*} = \pi_\sigma(\varphi^*) \tag{2.4}$$

Note that for any $\mu \in \mathscr{M}(\Lambda, \mathscr{G})$ there is $\nu \in \mathscr{M}(\Omega \times \Sigma_A, \sigma)$ such that $\Pi^*\nu = \mu$. (This is a well-known fact in the deterministic case. See ref. 8, Lemma 4.3, and we refer the reader to, for instance, Rudin, ref. 18, p. 112, for a proof of the Hahn–Banach theorem suitable for the modification. The proof for the random case is similar by using the corresponding facts presented in ref. 7.) This fact together with the variational principle implies that $\pi_\sigma(\varphi^*) \geqslant \pi_{\mathscr{G}}(\varphi)$ [this result can also be proved by arguments similar to the proof of ref. 8, Proposition 2.13, together with our (2.1)] and hence μ_φ is an equilibrium state of φ by (2.4). Moreover, by this fact together with (2.4) and Section 2.2 one easily sees that any equilibrium state of φ must be $\Pi^*\mu_{\varphi^*}$ and hence μ_φ. μ_φ is ergodic and strong-mixing because the bundle RDSs $(\mathscr{G}, \mu_\varphi)$ and $(\sigma, \mu_{\varphi^*})$ are isomorphic. ∎

3. EQUILIBRIUM STATE OF $\varphi = \varphi^{(u)}$

3.1. Existence and Uniqueness of Equilibrium State of $\varphi = \varphi^{(u)}$

In this section we will assume that Λ_0 is an Axiom A basic set of $f \in \mathrm{Emb}^2(O, M)$ and $\mathscr{U}(f)$ will be an open neighborhood of f in $\mathrm{Emb}^2(O, M)$. Fix arbitrarily $\lambda \in (\lambda_0, 1)$ and $\gamma \in (0, \gamma_0)$ and, correspondingly, let $\mathscr{U}(f)$ be given so that Proposition 1.3 holds true. For $\omega \in \Omega$ and $x \in \Lambda_\omega$, define

$$\varphi_\omega^{(u)}(x) = -\log |\det(T_x g_0(\omega)|_{E^u_{(\omega, x)}})|$$

By Proposition 1.3 (3), $\varphi^{(u)} := \{\varphi_\omega^{(u)} \in C(\Lambda_\omega)\} \in L^1_\Lambda(\Omega, C(X))$ and is equi-Hölder continuous. Thus, by Theorem 2.1, there is a unique equilibrium state, written $\mu_{\varphi^{(u)}}$, of $\varphi^{(u)}$ with respect to the bundle RDS $\mathscr{G}$. $\mu_{\varphi^{(u)}}$ is ergodic, and it is moreover strong-mixing if $f|_{\Lambda_0}$ is topologically mixing.

Note that for $\mu \in \mathscr{M}_e(\Lambda, \mathscr{G})$, by the Oseledec multiplicative ergodic theorem,

$$\int \varphi_\omega^{(u)}(x) \, d\mu(\omega, x) = -\sum \text{positive Lyapunov exponents of } (\mathscr{G}, \mu)$$

So $\mu_{\varphi^{(u)}}$ is the unique element of $\mathcal{M}_e(\Lambda, \mathcal{G})$ such that

$$\pi_{\mathcal{G}}(\varphi^{(u)}) = \sup_{\mu \in \mathcal{M}_e(\Lambda, \mathcal{G})} \left\{ h_\mu(\mathcal{G}) - \sum \text{positive Lyapunov exponents of } (\mathcal{G}, \mu) \right\}$$

$$= h_{\mu_{\varphi^{(u)}}}(\mathcal{G}) - \sum \text{positive Lyapunov exponents of } (\mathcal{G}, \mu_{\varphi^{(u)}})$$

We will call $\mu_{\varphi^{(u)}}$ the *generalized SRB measure* of the bundle RDS $\mathcal{G}$. We also remark that, by Ruelle's inequality,[2] one has

$$\pi_{\mathcal{G}}(\varphi^{(u)}) \leqslant 0$$

3.2. Escape Rate of $\mathcal{G}$ from Neighborhoods of Λ_ω

In this subsection we keep the conditions of the last subsection. Here we consider the relationship between pressure $\pi_{\mathcal{G}}(\varphi^{(u)})$ and escape rate of $\mathcal{G}$ from neighborhoods of the random hyperbolic invariant sets Λ_ω. Given a small neighborhood V_ω of Λ_ω for each $\omega \in \Omega$, we define for $n \geqslant 0$

$$V_{\omega, n} = \{ y \in V_\omega : g_\omega^k y \in V_{\tau^k \omega}, k = 0, 1, ..., n \}$$

and by $m(\cdot)$ we will denote the Lebesque measure of M. Since we do not assume that Λ_0 is an attractor of f, some points will escape from $V_{\tau^k \omega}$ under actions of g_ω^k when $\mathcal{U}(f)$ and V_ω are sufficiently small. Concerning the escape rate we have the following result when V_ω is taken to be an r-neighborhood of Λ_ω, i.e.,

$$V_\omega = B(\Lambda_\omega, r) := \{ y \in M : d(y, \Lambda_\omega) < r \}$$

We will write

$$B(\Lambda_\omega, n, r) = \{ y \in M : g_\omega^k y \in B(\Lambda_{\tau^k \omega}, r), k = 0, 1, ..., n \}$$

Proposition 3.1. There exist an open neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ and a corresponding number $r_0 > 0$ such that for any $0 < r \leqslant r_0$

$$\lim_{n \to +\infty} \frac{1}{n} \int \log m(B(\Lambda_\omega, n, r))\, dP(\omega) = \pi_{\mathcal{G}}(\varphi^{(u)})$$

Remark 3.2. Under the conditions of Proposition 3.1, if U is an open neighborhood of Λ_0 such that

$$U \subset B(\Lambda_\omega, r_0)$$

for all $\omega \in \Omega$, then one has

$$\lim_{n \to +\infty} \frac{1}{n} \int \log m(U_{\omega,n})\, dP(\omega) \leqslant \pi_{\mathscr{G}}(\varphi^{(u)})$$

where

$$U_{\omega,n} = \{ y \in M : g_\omega^k y \in U, k = 0, 1, ..., n \}$$

and if, moreover, for some small $\varepsilon > 0$ there holds

$$B(\Lambda_\omega, \varepsilon) \subset U \subset B(\Lambda_\omega, r_0)$$

for all $\omega \in \Omega$, then

$$\lim_{n \to +\infty} \frac{1}{n} \int \log m(U_{\omega,n})\, dP(\omega) = \pi_{\mathscr{G}}(\varphi^{(u)})$$

This result was given by Ruelle[21] for the deterministic case, and such neighborhoods U exist when $\mathscr{U}(f)$ is sufficiently small.

In the rest of this subsection we address to the proof of Proposition 3.1. We first present two preliminary lemmas. For $\omega \in \Omega$, $x \in \Lambda_\omega$, $\varepsilon > 0$, and $n \geqslant 0$, put

$$B_\omega(x, n, \varepsilon) = \{ y \in M : d(g_\omega^k x, g_\omega^k y) < \varepsilon, k = 0, 1, ..., n \}$$

Lemma 3.3 (Volume lemma). There exists an open neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ which has the following property: For small $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$C_\varepsilon^{-1} \leqslant m(B_\omega(x, n, \varepsilon))\, |\det(T_x g_\omega^n|_{E^u_{(\omega,x)}})| \leqslant C_\varepsilon$$

for all $\omega \in \Omega$, $x \in \Lambda_\omega$, and $n \geqslant 0$.

The proof of the lemma is given in the Appendix.

Lemma 3.4. One can find a neighborhood W of Λ_0 with $\bar{W} \subset O$, a neighborhood $\mathscr{V}(f)$ of f in $C^1(O, M)$, and numbers $\alpha^* > 0$ and $L^* > 0$

such that, if $0 < \alpha \leqslant \alpha^*$ and $\omega \in \prod_{-\infty}^{+\infty} \mathscr{V}(f)$, then any α-pseudo-orbit of ω that lies in $\bar{W}$ can be $L^*\alpha$-traced by an orbit of ω.

The proof of the lemma is almost the same as that of Proposition 2.6 of ref. 12.

Proof of Proposition 3.1. First let $\mathscr{U}(f)$ be as given in Lemma 3.3 and fix arbitrarily small $\varepsilon > 0$. Let $\omega \in \Omega$. Put, for $n \geqslant 0$,

$$B_\omega(n, \varepsilon) = \bigcup_{x \in \Lambda_\omega} B_\omega(x, n, \varepsilon)$$

For small $\delta > 0$, let $F_{\omega, n, \delta}$ be a maximal (ω, n, δ)-separated subset of Λ_ω. Clearly

$$B_\omega(n, \varepsilon) \subset \bigcup_{y \in F_{\omega, n, \delta}} B_\omega(y, n, \delta + \varepsilon)$$

and by Lemma 3.3

$$m(B_\omega(n, \varepsilon)) \leqslant C_{\delta + \varepsilon} \sum_{y \in F_{\omega, n, \delta}} \exp(S_n \varphi^{(u)})_\omega (y)$$

On the other hand, for $\delta \leqslant \varepsilon$ one has

$$\bigcup_{y \in F_{\omega, n, \delta}} B_\omega\left(y, n, \frac{\delta}{2}\right) \subset B_\omega(n, \varepsilon)$$

and the term on the left-hand side of the above inclusion is a disjoint union. So, by Lemma 3.3,

$$C_{\delta/2}^{-1} \sum_{y \in F_{\omega, n, \delta}} \exp(S_n \varphi^{(u)})_\omega (y) \leqslant m(B_\omega(n, \varepsilon))$$

This together with Section 2.1 proves

$$\lim_{n \to +\infty} \frac{1}{n} \int \log m(B_\omega(n, \varepsilon))\, dP(\omega) = \pi_{\mathscr{G}}(\varphi^{(u)}) \tag{3.1}$$

Now, by Lemma 3.4, one can shrink $\mathscr{U}(f)$ (if necessary) and find some number $0 < r_0 \leqslant \alpha^*$ so that (3.1) holds true for all $0 < \varepsilon \leqslant L^* r_0$ and

$$B_\omega(n, r) \subset B(\Lambda_\omega, n, r) \subset B_\omega(n, L^* r)$$

for all $\omega \in \Omega$, $n \geqslant 0$, and $0 < r \leqslant r_0$. Then the desired conclusion follows from (3.1). ∎

3.3. Random Perturbations of Hyperbolic Attractors

In this subsection we assume that Λ_0 is a *hyperbolic attractor* of $f \in \mathrm{Emb}^2(O, M)$, i.e., Λ_0 is an Axiom A basic set of f and there exists an open neighborhood U of Λ_0 such that $f\bar{U} \subset U$ and $\bigcap_{n \geqslant 0} f^n U = \Lambda_0$. Such a neighborhood U is called a *basin of attraction* of Λ_0. If an open neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ is given sufficiently small, defining the *unstable manifold* of the corresponding bundle RDS $\mathscr{G}$ at $(\omega, x) \in \Lambda$ as

$$W^u(\omega, x) = \{ y \in O : d(g_\omega^{-n} x, g_\omega^{-n} y) \to 0 \text{ as } n \to +\infty \}$$

then $W^u(\omega, x)$ is the image of $\mathbf{R}^{k_u}$ ($k_u = \dim E^u$) under an injective immersion of class $C^{1,1}$ and

$$W^u(\omega, x) \subset \Lambda_\omega$$

for all $(\omega, x) \in \Lambda$. In this case, a measure $\mu \in \mathscr{M}(\Lambda, \mathscr{G})$ is called an *SRB measure* of $\mathscr{G}$ [over $(\Omega, \bar{\mathscr{B}}(\Omega), P, \tau)$] if its conditional measures on the unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these submanifolds (see ref. 13, Chapter VII or ref. 3 for a more precise definition). Given $r > 0$, put

$$B_{\Lambda, r} = \bigcup_{\omega \in \Omega} \{\omega\} \times B(\Lambda_\omega, r)$$

A family of functions $\varphi = \{\varphi_\omega : B(\Lambda_\omega, r) \to \mathbf{R}\}_{\omega \in \Omega}$ is said to be *equicontinuous* if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in B(\Lambda_\omega, r)$ with $d(x, y) < \delta$ implies $|\varphi_\omega(x) - \varphi_\omega(y)| < \varepsilon$ for all $\omega \in \Omega$. By $\mathscr{F}^1(B_{\Lambda, r})$ we will denote the set of all equicontinuous families $\varphi = \{\varphi_\omega : B(\Lambda_\omega, r) \to \mathbf{R}\}_{\omega \in \Omega}$ which are such that $(\omega, x) \mapsto \varphi_\omega(x)$ is measurable on $B_{\Lambda, r}$ and $\int \|\varphi_\omega\| \, dP(\omega) < +\infty$, where $\|\varphi_\omega\| := \sup_{y \in B(\Lambda_\omega, r)} |\varphi_\omega(y)|$.

Our main result of this subsection is the following:

Theorem 3.5. Assume that Λ_0 is a hyperbolic attractor of $f \in \mathrm{Emb}^2(O, M)$. If an open neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ is given sufficiently small, and correspondingly let the bundle RDS $\mathscr{G}$ over $(\Omega, \bar{\mathscr{B}}(\Omega), P, \tau)$ be as introduced before, then there exists a unique $\mu \in \mathscr{M}(\Lambda, \mathscr{G})$ which is characterized by each of the following properties:

(1) μ is an SRB measure of $\mathscr{G}$.

(2) $h_\mu(\mathscr{G}) = \int \chi(\omega, x) \, d\mu$, where $\chi(\omega, x)$ is the sum of positive Lyapunov exponents of $\mathscr{G}$ at $(\omega, x) \in \Lambda$.

(3) μ is an equilibrium state of $\varphi^{(u)}$ with respect to $\mathscr{G}$.

This μ is ergodic, and it is strong-mixing if, moreover, $f|_{\Lambda_0}$ is topologically mixing. Furthermore, it has the following genericity property: There exists $r>0$ such that for any $\varphi \in \mathscr{F}^1(B_{\Lambda,r})$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_{\tau^k\omega}(g_\omega^k y) = \int_\Lambda \varphi \, d\mu$$

for $P\times m$-a.e. $(\omega, y)\in B_{\Lambda,r}$.

Remark 3.6. Results similar to Theorem 3.5 above were earlier proved by Young[25] through a different approach. See also Kiffer[11] for a stochastic stability treatment of the related topic.

In order to prove the theorem, we first collect some facts concerning stable and unstable manifolds of the bundle RDS $\mathscr{G}$ in the following lemma. The proof of these facts goes along standard lines and is omitted here [see ref. 13, Chapters III and VII, for a reference, and keep (A.1) of the Appendix of this paper in mind].

Lemma 3.7. Let Λ_0 be as given in Theorem 3.5. Then there exists an open neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ together with numbers $a>0$, $K>0$, and $0<\lambda<1$ such that the following hold true:

(1) For each $0<\delta\leqslant a$ there exists a continuous family of C^1 embedded k_s-dimensional ($k_s=\dim E^s$) disks $\{W_\delta^s(\omega, x)\}_{(\omega,x)\in\Lambda}$ which has the following properties for each $(\omega, x)\in\Lambda$:

(i) $W_\delta^s(\omega,x)=\exp_x \mathrm{Graph}(h_{(\omega,x)}^s|_{\{\xi\in E_{(\omega,x)}^s:\,|\xi|<\delta\}})$, where

$$h_{(\omega,x)}^s\colon\quad \{\xi\in E_{(\omega,x)}^s\colon |\xi|<a\} \to E_{(\omega,x)}^u$$

is a $C^{1,1}$ map with $\mathrm{Lip}(h_{(\omega,x)}^s)\leqslant \frac{1}{2}$.

(ii) $g_0(\omega)\, W_\delta^s(\omega,x)\subset W_\delta^s(G(\omega,x))$.

(iii) $d(g_\omega^n x, g_\omega^n y)\leqslant \lambda^n d(x,y)$ for all $y\in W_\delta^s(\omega,x)$ and all $n\geqslant 0$.

(iv) $\{y\in M\colon d(g_\omega^n x, g_\omega^n y)\leqslant \delta/K,\ n\geqslant 0\}\subset W_\delta^s(\omega,x)$.

(2) The obvious counterpart of (1) for local unstable manifolds $\{W_\delta^u(\omega,x)\}_{(\omega,x)\in\Lambda}$ holds true. Moreover, $W_\delta^u(\omega,x)\subset\Lambda_\omega$ for all $(\omega,x)\in\Lambda$.

(3) For each $0<\delta\leqslant a$, the family of C^1 embedded disks $\{W_\delta^s(\omega,z)\}_{z\in W_\delta^u(\omega,x)}$ is absolutely continuous (see ref. 13, Chapter VII, for the definition) and $\bigcup_{z\in W_\delta^u(\omega,x)} W_\delta^s(\omega,z)$ is a neighborhood of x.

Proof of Theorem 3.5. The equivalence of (1) and (2) is proved in ref. 3. By Theorem 2.1, in order to prove the first part of the theorem, it is sufficient to show $\pi_{\mathscr{G}}(\varphi^{(u)})=0$ when $\mathscr{U}(f)$ is sufficiently small.

First let $\mathscr{U}(f)$ be given so that Proposition 3.1 and Lemma 3.6 hold true. Let r_0 be the corresponding number given by Proposition 3.1. We may assume that the number a given by Lemma 3.6 satisfies $2a \leqslant r_0$ and $B(\Lambda_0, 2a) \subset U_0$, where U_0 is given by Proposition 1.3.

For $0 < \delta \leqslant a$ and $\omega \in \Omega$, put $W^s_\delta(\Lambda_\omega) = \bigcup_{x \in \Lambda_\omega} W^s_\delta(\omega, x)$. If we denote $(..., f, f, f,...)$ by ω_0, then $\Lambda_{\omega_0} = \Lambda_0$. Since $W^s_{a/6K}(\Lambda_0)$ is a neighborhood of Λ_0, there exists $0 < \rho < a/3K$ such that $B(\Lambda_0, \rho) \subset W^s_{a/6K}(\Lambda_0)$. Now choose $0 < \varepsilon < \rho/4$ and shrink $\mathscr{U}(f)$, if necessary, such that Proposition 1.3 holds true for the $\varepsilon > 0$ given above. Then, by Proposition 1.3 and Lemma 3.6, one has

$$\begin{aligned} B\left(\Lambda_\omega, \frac{\rho}{4}\right) &\subset B\left(\Lambda_0, \frac{\rho}{2}\right) \subset H_\omega B(\Lambda_0, \rho) \\ &\subset H_\omega W^s_{a/6K}(\Lambda_0) \subset W^s_a(\Lambda_\omega) \\ &\subset \bigcap_{n \geqslant 0} B(\Lambda_\omega, n, 2a) \end{aligned}$$

for all $\omega \in \Omega$. By Proposition 3.1, one has

$$\pi_{\mathscr{G}}(\varphi^{(u)}) = 0$$

This proves the conclusions of Theorem 3.5, except for the last assertion. Define $r = \rho/4$. Since μ is ergodic and it is an SRB measure, the last conclusion follows from the Birkhoff ergodic theorem and Lemma 3.6 together with the fact $B(\Lambda_\omega, r) \subset W^s_a(\Lambda_\omega)$ for all $\omega \in \Omega$ (see ref. 25 or ref. 13, Chapter VII, for a detailed proof). ∎

Remark 3.8. The arguments in Sections 2 and 3 exhibit some applications of structural stability theory to the ergodic theory of bundle RDSs arising from random diffeomorphism-type perturbations of Axiom A basic sets. It seems that the structural stability results presented in Section 1 could also have some applications to the dimension theory of such bundle RDSs. For example, let the bundle RDS $\mathscr{G}$ be as introduced in the conditions of Section 3.1 and let $\{h_\omega\}_{\omega \in \Omega}$ be the corresponding family of conjugacy homeomorphisms. If $\mathscr{R}_0$ is a Markov partition of Λ_0, then $\mathscr{R}_\omega := h_\omega \mathscr{R}_0$ is a random Markov partition of Λ_ω and this gives a nice measurable partition $\mathscr{R} := \{\{\omega\} \times R_\omega : R_\omega \in \mathscr{R}_\omega, \omega \in \Omega\}$ of Λ. Moreover, by Theorem 1.1 and Lemma 3.7, h_ω preserves stable and unstable manifolds, i.e., $h_\omega(W^a(f, x) \cap \Lambda_0) = W^a(\omega, h_\omega x) \cap \Lambda_\omega$, $a = s, u$, $x \in \Lambda_0$ for each $\omega \in \Omega$ when $\mathscr{U}(f)$ is small. Though it is possible to prove a version of the Eckmann–Ruelle conjecture[4] for general RDSs, the technical details would be very complicated. However, by means of the partition $\mathscr{R}$ introduced above, it is plausible that one could generalize the conjecture to our present bundle RDS $\mathscr{G}$ with a substantially simpler proof. [The Eckmann–Ruelle conjecture says in the case of $\mathscr{G}$ that, for each $\mu \in \mathscr{M}_e(\Lambda, \mathscr{G})$, almost all its sample measures μ_ω are exact dimensional and their pointwise dimension is equal to the sum of their stable and unstable pointwise dimensions.] In the 2-dimensional case, it is also plausible to generalize Manning's[14] result to the case of $\mathscr{G}$ by making use of the same partition $\mathscr{R}$.

APPENDIX

Proof of Lemma 3.3. We follow the main idea of Section 3 of Qian and Zhang,[17] where a similar result is proved for a deterministic endomorphism by adapting Bowen and Ruelle's[9] original idea to that case. For our present case it is more convenient to avoid the use of local stable and unstable manifolds.

Now let $\mathscr{U}(f)$ be an open neighborhood of f in $\mathrm{Emb}^2(O, M)$ which has the following properties:

(i) Proposition 1.5 holds true for $\mathscr{U}(f)$ and for some given $\lambda \in (\lambda_0, 1)$ and $\gamma \in (0, \gamma_0)$. This implies that there exists a constant $K = K(\gamma)$ such that for any $\xi \in E_\Lambda$, $\xi = \xi^s + \xi^u \in E^s_\Lambda \oplus E^u_\Lambda$

$$\|\xi\| := \max\{|\xi^s|, |\xi^u|\} \leqslant K|\xi| \tag{A.1}$$

(ii) There exist $r_0 > 0$, $A_0 \geqslant 1$ such that for any $(\omega, x) \in \Lambda$

$$\tilde{g}_{(\omega, x)} := \exp^{-1}_{g_0(\omega)x} \circ g_0(\omega) \circ \exp_x \colon \{\xi \in T_x M \colon |\xi| \leqslant r_0\} \to T_{g_0(\omega)x} M$$
$$\tilde{g}^-_{(\omega, x)} := \exp^{-1}_x \circ g_0(\omega)^{-1} \circ \exp_{g_0(\omega)x} \colon \{\eta \in T_{g_0(\omega)x} M \colon |\eta| \leqslant r_0\} \to T_x M$$

are well defined,

$$\sup_{\xi \in T_x M,\, |\xi| \leqslant r_0} |T_\xi \tilde{g}_{(\omega, x)}|, \qquad \sup_{\eta \in T_{g_0(\omega)x} M,\, |\eta| \leqslant r_0} |T_\eta \tilde{g}^-_{(\omega, x)}| \leqslant A_0 \tag{A.2}$$

and

$$\mathrm{Lip}(T.\tilde{g}_{(\omega, x)}), \qquad \mathrm{Lip}(T.\tilde{g}^-_{(\omega, x)}) \leqslant A_0 \tag{A.3}$$

where the Lipschitz constants are taken with respect to $|\cdot|$.

Let $k_u = \dim E^u$ and $A_1 = 4k_u(2A_0)^{k_u}$. Fix arbitrarily $\varepsilon_0 > 0$ so that $\varepsilon_0 < \min\{\frac{1}{2}(1-\lambda), \frac{1}{2}(\lambda^{-1}-1)\}$ and $(\lambda+\varepsilon_0)^{-1} - \varepsilon_0 > 1$. Define $\mu = (\lambda + 2\varepsilon_0)[(\lambda+\varepsilon_0)^{-1} - \varepsilon_0]^{-1} < 1$. Fix arbitrarily $\delta_0 \in (0, 1)$ so that $e^{-2\delta_0} < 1 - \delta_0$. Finally, choose $r > 0$ such that

$$r \leqslant \min\left\{r_0, \frac{\varepsilon_0}{4KA_0}, (1-\mu)\frac{\delta_0}{A_0 A_1 K}\right\} \tag{A.4}$$

and for any $(\omega, x) \in \Lambda$

$$\mathrm{Lip}(R_{(\omega, x)}), \quad \mathrm{Lip}(R^-_{(\omega, x)}) \leqslant \frac{\varepsilon_0}{2K} \tag{A.5}$$

where $R_{(\omega, x)} := (\tilde{g}_{(\omega, x)} - T_0\tilde{g}_{(\omega, x)})|_{\{\xi \in T_xM:\, |\xi| \leqslant r\}}$, $R^-_{(\omega, x)}$ is defined analogously, and the Lipschitz constants are taken with respect to the norm $|\cdot|$.

Define for $\omega \in \Omega$, $x \in \Lambda_\omega$, and $n \geqslant 0$

$$D_\omega(x, n, r) = \{\xi \in T_xM: |\tilde{g}^k_{(\omega, x)}\xi| \leqslant r, k = 0, 1,..., n\}$$

where $\tilde{g}^k_{(\omega, x)} := \tilde{g}_{G^{k-1}(\omega, x)} \circ \cdots \circ \tilde{g}_{(\omega, x)}$. Now we give some conclusions (claims).

Claim A.1. If $\xi \in D_\omega(x, n, r)$, then

$$|\tilde{g}^k_{(\omega, x)}\xi| \leqslant 2Kr \max\{\alpha^k, \alpha^{n-k}\}$$

for $k = 0, 1,..., n$, where $\alpha = (\lambda^{-1} - \varepsilon_0)^{-1}$.

The proof of this claim is the same as that of Lemma 1.2.

Let $\omega \in \Omega$, $x \in \Lambda_\omega$ and $\xi \in T_xM$ with $|\xi| \leqslant r$. Assume that V is a subspace of T_xM with $V \oplus E^s_{(\omega, x)} = T_xM$ and let $L_V: E^u_{(\omega, x)} \to E^s_{(\omega, x)}$ be the linear map such that $V = \mathrm{Graph}(L_V)$. Write $\theta_0(V) = |L_V|$ and $\theta_k(\xi, V) = \theta_0(T_\xi \tilde{g}^k_{(\omega, x)} V)$, $k > 0$, if they are well defined.

Claim A.2. If $\theta_0(V) \leqslant 1$, then $\theta_1(\xi, V)$ is well defined and

$$\theta_1(\xi, V) \leqslant \mu\theta_0(V) + KA_0 |\xi|.$$

This claim follows from (A.2)–(A.4) and the standard techniques of graph transformation (see, for instance, ref. 13, Proposition VII.2.1).

Claim A.3. If $\theta_0(V) \leqslant A_1^{-1}\delta_0$, then

$$e^{-A_1(\theta_0(V) + |\xi|)} \leqslant \left|\frac{\det(T_\xi \tilde{g}_{(\omega, x)}|_V)}{\det(T_0\tilde{g}_{(\omega, x)}|_{E^u_{(\omega, x)}})}\right| \leqslant e^{A_1(\theta_0(V) + |\xi|)}$$

The proof of this claim is the same as that of ref. 17, Lemma 3.4.

Using Claims A.1–A.3, one can prove the following by a simple computation.

Claim A.4. There exists a constant $A_2 > 0$ such that for all $(\omega, x) \in \Lambda$ and $n \geqslant 0$

$$A_2^{-1} \leqslant \left| \frac{\det(T_\xi \tilde{g}^n_{(\omega, x)} | V)}{\det(T_0 \tilde{g}^n_{(\omega, x)} | E^u_{(\omega, x)})} \right| \leqslant A_2$$

if $\xi \in D_\omega(x, n, r)$ and $\theta_0(V) \leqslant A_1^{-1} \delta_0$. With other things being fixed, A_2 can be taken arbitrarily close to 1 when r and $\theta_0(V)$ are sufficiently small.

By standard techniques of graph transformation (see, for instance, ref. 12) together with Claim A.2 one can prove the following:

Claim A.5. Let $0 < \rho \leqslant r/2$ be given. For $(\omega, x) \in \Lambda$, define $B_{(\omega, x)}(\rho) = B^s_{(\omega, x)}(\rho) \times B^u_{(\omega, x)}(\rho)$, where $B^a_{(\omega, x)}(\rho) = \{\xi \in E^a_{(\omega, x)} : |\xi| < \rho\}$, $a = s, u$. If $h: B^u_{(\omega, x)}(\rho) \to B^s_{(\omega, x)}(\rho)$ is a C^1 map with $\mathrm{Lip}(h) \leqslant A_1^{-1} \delta_0$, then there is a C^1 map $k: B^u_{G(\omega, x)}(\rho) \to B^s_{G(\omega, x)}(\rho)$ with $\mathrm{Lip}(k) \leqslant A_1^{-1} \delta_0$ such that

$$[\tilde{g}_{(\omega, x)} \, \mathrm{Graph}(h)] \cap B_{G(\omega, x)}(\rho) = \mathrm{Graph}(k)$$

In view of (A.1), there is the following:

Claim A.6. Let $0 < \rho \leqslant r/2$. There is a constant $K_\rho > 0$ such that for all $(\omega, x) \in \Lambda$ one has

$$K_\rho^{-1} \leqslant m_{x, h}(\mathrm{Graph}(h)) \leqslant K_\rho$$

if $h: B^u_{(\omega, x)}(\rho) \to B^s_{(\omega, x)}(\rho)$ is a C^1 map with $\mathrm{Lip}(h) \leqslant A_1^{-1} \delta_0$, where $m_{x, h}$ denotes the Lebesque measure on $\mathrm{Graph}(h)$ induced by its inherited Riemannian metric as a submanifold of $T_x M$ (associated with the original Riemannian scalar product).

Now let $0 < \rho \leqslant r/2$ be given. Fix arbitrarily $(\omega, x) \in \Lambda$ and $n \geqslant 0$. Define for $\xi^s \in B^s_{(\omega, x)}(\rho)$

$$C(\xi^s, n, \rho) = \{\xi \in T_x M : \xi - \xi^s \in E^u_{(\omega, x)}, \|\tilde{g}^k_{(\omega, x)} \xi| < \rho, k = 0, ..., n\}$$

By Claim A.5, $\tilde{g}^n_{(\omega, x)} C(\xi^s, n, \rho)$ is the graph of a C^1 map $h_{\xi^s, n}: B^u_{G^n(\omega, x)}(\rho) \to B^s_{G^n(\omega, x)}(\rho)$ with $\mathrm{Lip}(h_{\xi^s, n}) \leqslant A_1^{-1} \delta_0$. Hence, by Claim A.6,

$$\begin{aligned} K_\rho^{-1} &\leqslant m_{g^n_\omega x, h_{\xi^s, n}}(\tilde{g}^n_{(\omega, x)} C(\xi^s, n, \rho)) \\ &= \int_{C(\xi^s, n, \rho)} |\det(T_\xi \tilde{g}^n_{(\omega, x)} | E^u_{(\omega, x)})| \, dm_{x, h_{\xi^s, 0}} \leqslant K_\rho \end{aligned}$$

where $h_{\xi^s,0}: B^u_{(\omega,x)}(\rho) \to B^s_{(\omega,x)}(\rho)$, $\eta^u \mapsto \xi^s$. This together with Claim A.4 implies that

$$(K_\rho A_2)^{-1} \leqslant m_{x, h_{\xi^s,0}}(C(\xi^s, n, \rho)) \, |\det(T_x g^n_\omega|_{E^u_{(\omega,x)}})| \leqslant K_\rho A_2$$

Therefore, writing $N_\omega(x, n, \rho) = \{\xi \in E_{(\omega,x)}: \|\tilde{g}^k_{(\omega,x)}\xi\| < \rho,\ k = 0,..., n\}$, one has by the Fubini theorem

$$(C'_\rho)^{-1} \leqslant m_x(N_\omega(x, n, \rho)) \, |\det(T_x g^n_\omega|_{E^u_{(\omega,x)}})| \leqslant C'_\rho$$

since $N_\omega(x, n, \rho) = \bigcup_{\xi^s \in B^s_{(\omega,x)}(\rho)} C(\xi^s, n, \rho)$, where $m_x(\cdot)$ denotes the Lebesgue measure on $T_x M$ associated with the Riemannian scalar product and C'_ρ is a number depending only on $\mathscr{U}(f)$ and ρ.

Let $0 < \varepsilon < r/2K$. We obtain the conclusion of Lemma 3.3 by

$$\exp_x N_\omega\left(x, n, \frac{\varepsilon}{2}\right) \subset B_\omega(x, n, \varepsilon) \subset \exp_x N_\omega\left(x, n, \frac{r}{2}\right)$$

for all $(\omega, x) \in \Lambda$ and $n \geqslant 0$. ∎

In a similar way one can prove the following:

Lemma A.1. (Second volume lemma). There exists an open neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Emb}^2(O, M)$ which has the following property: For small $\varepsilon, \delta > 0$ there is a constant $C_{\varepsilon,\delta} > 0$ such that

$$C^{-1}_{\varepsilon,\delta} \leqslant \frac{m(B_\omega(y, n, \delta))}{m(B_\omega(x, n, \varepsilon))} \leqslant C_{\varepsilon,\delta}$$

whenever $\omega \in \Omega$, $x \in \Lambda_\omega$, $n \geqslant 0$, and $y \in B_\omega(x, n, \varepsilon)$.

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